

CHAPTER 9

Fluids

9.1 INTRODUCTION

Microfluidics involves the design and the application of technical systems which manipulate fluids at the microscale. As we will see, numerous effects can be exploited at this length scale that cannot (or hardly) be seen in fluidic systems at greater length scales. As an example, most microfluidic systems show strictly laminar flow behavior. This gives rise to numerous effects, *e.g.*, filtering and laminar flow stacking, which are very difficult to obtain in systems at greater length scales.

Scaling Law. One important advantage of microfluidics is often referred to as the *scaling law*. If a system is decreased in size linearly in all dimensions, the volume of the system will shrink with the power of three. As an example, if an analytical system working on an analyte provided as cubic volume with an edge length of 1 mm, is reduced in lateral size by the factor of 10, the total volume required will shrink from 1 mm³, which is 1 μ l to 1 nl. This is a drastic decrease in liquid demand.

Different Dominating Effects. In a microfluidic system, convection is almost nonexistent and therefore not an important process in transport phenomena of, *e.g.*, momentum or mass. In contrast, diffusion has a significant impact for the performance of a microfluidic system. In macroscopic fluidic systems, convection is the dominating transport process, whereas diffusion can usually be neglected. In macroscopic fluidic systems (*e.g.*, a bathtub), we usually consider surface effects to be of minor importance, which are of paramount importance in microfluidics due to the high surface-to-volume ratios. Gravity is another example. Macroscopic fluidic systems are strongly influenced by gravity. In microfluidic systems, gravity is usually only considered for fluidic storage vessels and when calculating gravitational pressure drops that can drive microfluidic flows. Surface tension is another important effect for microfluidics that is rarely considered on the macroscopic scale.

9.2 SOLIDS, LIQUIDS, AND GASES AT THE ATOMIC SCALE

Critical Shear Stress. Before discussing the mechanics of fluids, let us briefly discuss what exactly is to be understood by this term. There are many different definitions for the term “fluid,” many of which refer to the fact that a fluid can be irreversibly deformed even by small shear stresses. This behavior is somewhat different from the behavior of solids, which will deform elastically if a certain critical shear stress τ_{crit} is not exceeded (see Fig. 9.1a). If this critical threshold was not exceeded, the solid will regain its original shape upon release of the shear stress. However, if the critical shear stress was exceeded, the solid will deform inelastically (see Fig. 9.1b). Upon release, the original shape is not regained and the solid remains deformed. In contrast to solids, fluids will change their shape even if the applied shear stresses are very small (*e.g.*, stress exerted by the effect of gravity). The fluid will deform in order to reduce or balance this stress and thus adapt its shape. If the shear stress is released, the fluid will not change its shape.

Solids. This difference in behavior can be explained simply by looking at solids and fluids on an atomic scale (see Fig. 9.2). In solids, the atoms are packed densely, forming regular patterns or lattices. These regular patterns are held in place by intermolecular forces. The atoms oscillate about their origin due to Brownian motion. However, they cannot move freely, only vibration motion is possible. Their kinetic energy is small compared to their thermal energy, and individual atoms interact strongly with each other. Upon application of external

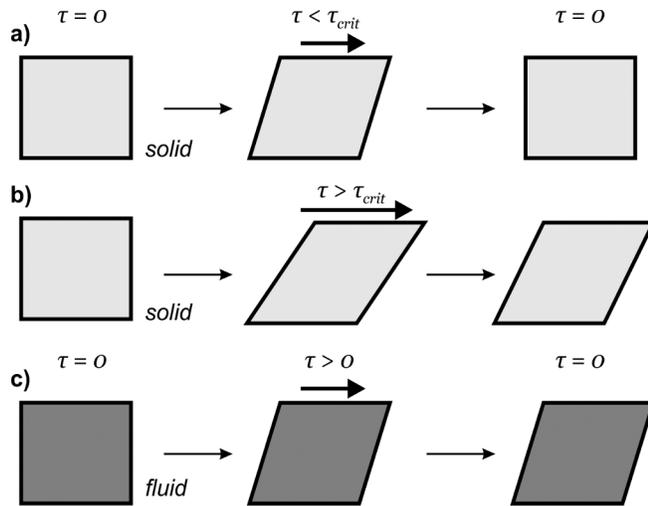


Fig. 9.1 Shear stress on solids and liquids. a) A solid will deform elastically if a shear stress below a certain critical threshold shear stress τ_{crit} is applied. Upon release the solid will regain its original shape. b) If a shear stress above the critical threshold τ_{crit} is applied, the solid is inelastically deformed. Upon release of the shear stress, the solid will not regain its original shape and will remain deformed. c) Fluids have very low critical shear stresses, which results in the fact that they will change their shape even if very small shear stresses are applied. This allows a fluid to adapt its shape, e.g., to the container holding the fluid.

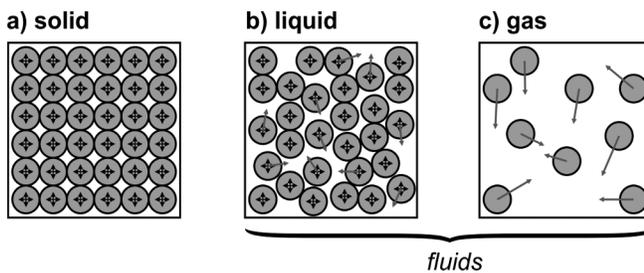


Fig. 9.2 Schematic of solids (a), liquids (b), and gases (c) on the atomic level. The molecules oscillate around their origin by Brownian motion. Due to the dense packing, atoms in solids cannot move freely. In liquids, the atoms are able to move in addition to oscillation movement about their origin. In gases, the kinetic energy is dominating and the thermal oscillation is only of minor importance. This is why liquids and gases (both of which are classified as fluids) change shape easily and do not regain their original shape after application of shear stress.

stresses, this regular structure is distorted, which gives rise to significant elastic forces that make a solid retain its original shape once the external stress is released. If the applied shear stress is above a critical shear stress, the structure will be deformed inelastically, which results in a solid not regaining its original shape after release of the applied stress. If solids are heated, the vibration increases until eventually the vibration motion exceeds the retaining intermolecular forces thus breaking the regular structure. The solid then transits to the liquid state (see Fig. 9.2b).

Liquids. In liquids, the distance between the individual atoms is bigger. Liquids do not form regular patterns and the particles are free to move to a certain extent, in addition to vibration motion. Upon application of external stress, the atoms are free to move, which changes the shape of the liquids and reduces the effect of the external stress. This is why a liquid will deform even upon application of only small shear stresses. In gases, this effect is even more pronounced. Liquids, similar to solids, expand upon heating due to the increased oscillation motion which spaces the particles farther, thus allowing enhanced movement. Liquids usually have similar densities as solids. If heated above a critical temperature, a liquid will transition to the gas phase (see Fig. 9.2c).

Gases. In gases, the individual particles will break all short-range interaction. The particles can move freely and interact only *via* direct collision. As can be seen, the overall density of gases is lower compared to liquids and solids which facilitates particle rearrangement due to, e.g., compression. Gases are compressible fluids whereas most liquids can be considered *non-compressible* or *incompressible*. In gases, there are significant void spaces between the individual atoms. This can also be seen when looking at the differences in density of a substance in liquid and gaseous form. As an example, the liquid density of water is around 1.0 g cm^{-3} whereas the density of water vapor is around 0.8 g dm^{-3} which is three orders of magnitude smaller.

Phase Changes. Many substances exist either in solid, liquid, or gaseous forms or states, depending on the conditions (temperature, pressure). These states are often referred to as *phases*, the transition between them is referred to as *phase change* or *phase transition*. These usually have fixed technical terms

- from solid to liquid: *melting*
- from liquid to solid: *freezing*

- from liquid to gaseous: *vaporization*
- from gaseous to liquid: *condensation*
- from solid to gaseous: *sublimation*
- from gaseous to solid: *resublimation* or *deposition*

Plasmas. Besides these three phases, there is a fourth state in which matter exists: the plasma state. This is by far the most common state of matter in the universe, as almost all stars, including our sun, consist mainly of plasma. On earth, plasmas can be found, *e.g.*, during lightning effects. Plasmas can be created at high electric field strengths that are sufficient to strip electrons of atoms, leaving the atoms as positive ions separated from their electrons. Plasmas usually have high charge carrier densities (positively charged ions, cations, and negatively charged electrons) which makes them electrically conductive. Plasmas are usually created from gases that are submitted to high electric fields. The transition from the gaseous to the plasma state is referred to as *ionization*. The transition from plasma to gaseous state occurs if electrons and ions are left to recombine, which is why this transition is termed *recombination*.

9.3 CONTROL VOLUMES

Compared to solids, the numerical and analytical characterization of fluids seems less intuitive. After all, a solid has a boundary that defines the outer (physical) contour. Given the higher mobility of their individual molecules, fluids (both liquids and gases) usually fill provided restricting volumes (such as a vessel or containers). As the outer shape changes with the restricting volume, fluids do not show distinct physical contours. Nevertheless, analytical and numerical solutions can only be derived, if suitable control volumes are defined. For static fluids, such control volumes could be adopted from the restricting volumes - but what if the dynamic case of a flowing liquid is to be considered? As we will see, the choice of control volume is tightly linked to the problem at hand, *i.e.*, which length scale is to be considered. The highest level of resolution is achieved if individual molecules are considered.

9.3.1 Lennard-Jones Potential

At the highest level of details, a fluid is approximated as a large collection of individual particles, each of which represents a single molecule. The interaction of each of these particles with its surrounding will allow the analytical description of the bulk fluid. The particles interact by forces which act at different length scales (primary and secondary interaction). Instead of forces, the acting potential of the individual particles is studied from which the forces may be derived. In this case, the most common approach is modeling of the atom interaction using the so-called *Lennard-Jones potential*.

When two atoms are brought into close contact, there will be a certain repulsive force acting due to the fact that the orbitals of the two atoms will overlap. In simpler words, the electrons of each atom will repulse the atoms of the other atom, respectively. The same holds true for the positively charged nuclei. In parallel the nucleus at one atom will attract the electron of the second atom and \rightarrow . This gives rise to attractive forces. Both of these forces are functions of the distance r of the two atoms (see Fig. 9.3). The action of these forces is described analytically by the Lennard-Jones potential which was described by British mathematician John Lennard-Jones in 1924 [1, 2] as

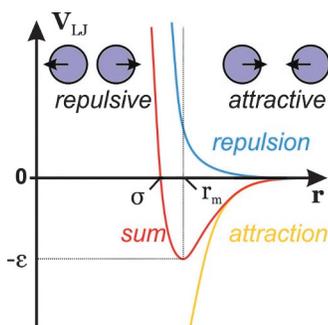


Fig. 9.3 Lennard-Jones potential given as a function of the distance of two rigid particles. The particles transition from predominantly repulsive interaction to an attraction regime if the distance increases over a critical distance to r_m . The interaction forces decrease exponentially at greater distance.

Tab. 9.1 Some values for calculating the Lennard-Jones potential. Taken from [3]

Substance	σ [Å]	$\frac{\epsilon}{k_B}$
hydrogen gas	2.915	38.0
air	3.617	97.0
nitrogen gas	3.667	99.8
oxygen gas	3.433	113.0
carbon monoxide	3.590	110.0
carbon dioxide	3.996	190.0
methane	3.780	154.0
benzene	5.433	387.0

$$\begin{aligned}
 V_{LJ} &= 4\epsilon \left(\left(\frac{\sigma}{r} \right)^n - \left(\frac{\sigma}{r} \right)^6 \right) \\
 &= \epsilon \left(\left(\frac{r_m}{r} \right)^n - 2 \left(\frac{r_m}{r} \right)^6 \right)
 \end{aligned}
 \tag{Eq. 9.1}$$

where ϵ is the depth of the potential well, r_m is the distance for which the Lennard-Jones potential is 0, and σ is the distance for which the Lennard-Jones potential is minimal, *i.e.*, the bond length. The Lennard-Jones potential describes the repulsive and attractive forces in potential form as a function of the distance of the particles. The acting forces may be derived as the first derivative

$$F_{LJ}(r) = \frac{dV_{LJ}}{dr}$$

The model is the basis of most modern computer-aided molecular modeling. In order to make it computationally less expensive to apply, the exponent n is usually chosen to be $n = 12$, in which case the first term is simply the second-term square. The model can be fitted with high accuracy to experimental results that makes the model a useful tool in modern molecular modeling and quantum chemistry, even though more precise models exist for some applications (which usually are computationally more expensive). The constants for various gases can be obtained from textbooks. Tab. 9.1 lists a selection of constants for commonly used substances.

The main drawback of the Lennard-Jones model is the fact that it does not support directionality, as the interaction is modeled solely on the distance variable r . For directed bonds (such as covalent bonds) the model fails to precisely model the interaction, especially if two atoms are connected by multiple bonds.

The Lennard-Jones model thus predicts that two atoms, once brought into sufficiently close contact, will experience a potential minimum and thus will remain in close contact. In this state, an equally strong repulsive and attractive force is present and the atoms stay in close proximity. This is the bonded state.

In many applications in fluid mechanics, modeling the behavior of a fluid on the scale of individual atoms is impractical, as this involves significant computational effort. On modern computer systems, numerical approximation of only a few thousands to ten thousands of molecules over a short-time scale is computationally feasible. More complex scenarios or larger control volumes require the usage of fast clusters and intelligent approaches in exploiting parallelization effects or restricting interaction lengths. One potential simplification would be restricting the interaction lengths to a few nm instead of calculating the interaction effect of particles that are several mm apart. However, modeling on this length scale will (for at least another decade) be restricted to characterization of small fluid volumes in the range of nm or fractions thereof.

9.3.2 Continuum Hypothesis

In most cases in fluid mechanics, the smallest entity considered is a control volume of suitable size (usually in the range of several μm edge length). This control volume contains a significant number of individual atoms and thus averages all effects exerted by the individual atoms. Usually this averaging results in a significantly more predictable behavior. In this case, we treat the fluid as being a continuous piece of matter, which is why this approach is referred to as the *continuum approach* or the *continuum hypothesis*. This continuum can be treated as having average values for velocity, acceleration, entropy, or enthalpy. However, these are averaged values, so

they are discontinuous when looking at the atomic scale. On the experimental scale, observing the effects in a continuum gives rise to more steady experimental data. If the resolution is increased, the measurements will become less steady as the effects of the individual atoms become more and more pronounced. The continuum hypothesis is applicable for most applications in classical macro- and microfluidic applications as the size of the control volumes can be chosen sufficiently big to contain at least some 10 000 atoms. In this case, the statistical variations are roughly in the range of 0.1 %. As we will see the length scales required for this prerequisite are in the range of about 100 nm to 1 μm for gases (see section 9.3.4) and 10 nm for liquids (see section 9.3.3). If the scale length of the fluidic system is decreased, the continuum hypothesis may not be applicable. As an example, discussing the flow of liquids in a nanopore, the control volume will contain only a few atoms that may not be treated as a homogeneous continuum.

The exact dimensions of the control volumes must be adapted to the question in mind: What level of detail is required? The bigger the control volume, the less expensive the calculation will be numerically. However, increasing the control volume size above a critical threshold may result in loss of detail on the molecular length scale (if, *e.g.*, mass transport phenomena such as diffusion are to be considered). A suitable length scale is found once a control volume does not show significant signs of fluctuation on the molecular level, and all molecules in the control volume may be approximated as having the same field variables values (such as temperature or pressure, see section 10.1). In this state, the fluid behaves as a continuum. A fluid continuum may be approximated as being composed of individual control volumes, each of which has distinct values of the field variables.

9.3.3 Number of Water Molecules in a Control Volume of 10 nm Edge Length

As an example, let us consider the number of molecules in a cubic control volume of 10 nm edge length. Water has a density of 1 g ml^{-1} and a molecular weight of 18 g mol^{-1} . Thus, 18 g of water occupy a control volume of 18 ml and contain 1 mol substance. From Eq. 6.1 we know that there are 6.022×10^{23} molecules in 1 mol substance. Therefore we can calculate the volume of a single water molecule to be 0.0299 nm^3 (assuming a cubic form), which amounts to an edge length of around 0.31 nm. This can roughly be estimated to be the diameter of a single water molecule. Therefore we find that a cubic control volume of 10 nm edge length will contain 33 455 molecules. This number is sufficiently large to treat the whole volume as having homogenous fluid properties (velocity, enthalpy, entropy, etc.) and thus apply the continuum hypothesis.

9.3.4 Number of Gas Molecules in a Control Volume of 100 nm Edge Length

Extending the example, we now turn to calculating the number of molecules of a gas in a control volume of a given size. This calculation holds true for all gases in equilibrium. We know that in equilibrium, an ideal gas will occupy a molar volume of 22.414 L (see section 6.4.1). Again we can calculate that one gas molecule will occupy a total volume of 37.22 nm^3 which amounts to a cube with an edge length of 3.34 nm. This value is two orders of magnitude bigger than the values found for liquids, which explains the significant difference in compressibility between gases and liquids. Using these values, it can be found that a cube with edge length of 10 nm will hold a total of 26 molecules only. A cube of 100 nm edge length will hold around 33 450 molecules which is above the threshold. Often, for gases an edge length of 1 μm is suggested, which amounts to 3.35×10^{10} molecules.

9.3.5 Dense and Dilute Gases: Number of Gas Molecules in a Control Volume of 100 nm Edge Length at a Pressure of 200 bar

The calculations made in section 9.3.4 refer to gases under standard conditions, *i.e.*, 1 bar pressure. If the container was pressurized, the calculation must be slightly modified. For this, we consider a bottle of nitrogen gas with a volume of 20 L that is filled up to a pressure of 200 bar. The temperature of the bottle is 273.15 K. The amount of substance in the bottle can be calculated using the ideal gas law (see Eq. 6.4) to be 178.46 mol. From this, the average volume a single molecule occupies can be calculated to be 0.18 nm^3 with an edge length of 0.57 nm. Using these values, it can be found that a cube with edge length of 10 nm will hold a total of 5373 molecules, a cube with 100 nm edge length around 5.37×10^6 molecules.

From this calculation, we can derive an important characteristic of gases, which is referred to as the *dilution factor* δ that is defined as

$$\delta = \frac{l_{\text{STP conditions}}}{l_{\text{given conditions}}}$$

which puts the average molecule distance at standard conditions (around 3.34 nm for most molecules) in relation to the spacing at the given conditions. In our example, the dilution factor would amount to $\delta = \frac{3.34 \text{ nm}}{0.57 \text{ nm}} = 5.86$. Gases with $\delta > 1$ are referred to as *dense gases*, whereas gases with $\delta < 1$ are referred to as *dilute gases*. In dilute gases, collision is the only dominating interaction effect. As we can see, the continuum assumption is easier fulfilled for dense gases than for dilute gases.

9.4 FLUID PROPERTIES

In terms of fluid mechanics, fluids have distinct properties. They may be classified as being either kinematic properties (related to the movement of the fluid), thermodynamic properties (related to the energy transport from one control volume to the next), and transport properties (related to the ability of the fluid to transfer certain properties from one control volume to the next).

Kinematic Properties. These are the properties that can be derived from the movement of the fluid. The most important kinematic property of a fluid is the velocity \vec{v} that describes the flow field. This is also the only vector field variable (see section 10.1.1) of interest. The kinematic properties are imposed upon the fluid by external forces (pressure drop, gravity, *etc.*) and are therefore not inherent properties of the fluid's material. A certain restricting volume (such as a technical pipe or a microfluidic channel) will enforce a certain flow pattern, in many cases irrespective of the actual fluid being probed through it.

Thermodynamic Properties. Besides the kinematic properties, fluids have several thermodynamic properties: pressure p , temperature T , density ρ , enthalpy H , and entropy S . As we will see in section 10.1.1, pressure, temperature, and density are field variables that must be known in order to characterize the flow field. Enthalpy and entropy are usually expressed as functions of these field variables.

Transport Properties. Fluids can transport momentum (by means of viscosity), heat (by conduction), and mass (by diffusion). As these properties are inherent to the fluid they are generally referred to as the fluid's "transport properties." The most important transport properties of fluids are discussed in the following section.

9.5 MOMENTUM TRANSPORT

The capability to transport momentum is one of the most important properties of fluids. As discussed, a fluid can be thought of as a collection of molecules that interact. If momentum is transferred from one molecule, this interaction will result in the transport of this momentum onto molecules in close proximity. Therefore, the fluid transports this momentum through the volume. Momentum transport is often referred to as *fluid friction*. If subjected to an external force or momentum, a fluid layer will move along the direction of the momentum. While moving, the fluid layer drags adjacent fluid layers along, coupled by the friction forces acting between them. The behavior is similar to a deck of cards: If the topmost card is moved, friction between the cards will drag cards further down the deck along. This allows, *e.g.*, spreading or fanning out of card decks. In a simple experiment, the fluid layer between two plates may be compared to a deck of cards which is placed in between a surface (such as a table) and the hand of the card dealer. The topmost card (in this case, the top plate) is displaced, thus a force acts upon the topmost card which moves in consequence. The topmost card transfers the movement applied to the card located directly underneath. As a result, this card will also be dragged along. This effect occurs due to friction forces acting between the two cards. This card will, again, transfer momentum to the card located underneath and so forth. As a consequence, the card deck will be displaced, which allows the described fanning.

A Word on Momentum. Many students have a little difficulty grasping the concept of momentum. Most of us know *Newton's cradle* and we know that the way the small balls are set in motion depends on the number of balls we initially displaced. We also know that this is related somehow to momentum transport. But what exactly is momentum? The concept of momentum is inherently linked to the concept of forces. Forces cannot be visualized, but we can see their action. If we drop a pen we cannot see gravity pulling it to the floor, but we can see the effect as the pen falls. So we know there must be a force acting.

You can think of momentum as being essentially this “action of forces.” In physics we usually use forces when setting up a force balance, *i.e.*, in order to determine which force is acting on a *mass* at a given moment in our system. There you can also see the second important component on which momentum is calculated: mass. Momentum is calculated as the product of mass and velocity and therefore given by

$$p = mv$$

Momentum therefore relates to the action that a force has exerted on a given mass. Or in other words, momentum “stores” the effect of a force on a mass. This force may long have been removed from a system, but its action is still there. If a force has accelerated a mass, the mass has a given momentum. We can always tell that there, in fact, was a force acting on the mass because it has momentum. Momentum *conserves* the action of forces. This is what makes momentum so useful in practice. It is a *conservative variable*, *i.e.*, we can setup conservation equations just as we can do with energy. Obviously, this is significantly easier than working with the forces directly. Let us consider a mass upon which an accelerating force acted for only a fraction of a second. We are interested in the way this mass moves for the next 10 s. How are we to set up these equations? The force only acted for a moment, but for how long exactly? How finely do we have to resolve the time space in order to catch the action of the force in its entirety? The answer to this question is simple: We don’t. We simply assume that at the beginning of our experiment, the force has already acted upon the mass and stored its effect in the form of momentum. We then proceed by assuming that other forces (*e.g.*, damping) acts upon the mass, slowly draining the momentum until the mass comes to a standstill. Suddenly, the whole physics has become significantly easier.

Therefore, whenever you see forces in action, we usually refrain from setting up balances of forces, but rather work with the momentum these forces create. As stated, momentum is a conservative variable which means that many concepts we know from similar conservative variables (such as energy or mass) can be transferred directly to momentum. Transport phenomena are good examples. Mass can be transport, *e.g.*, by diffusion or convection. However, irrespective of the transport mechanism, mass must be conserved. Within a control volume, we can balance the in- and outflux of mass and therefore state the change of mass over time within the control volume. We can do the same thing with energy. Again, heat is a conservative variable, therefore we can set up conservative forms of our heat equations. The same applies for momentum. Momentum is also a conservative variable, therefore we can setup conservative forms. The Navier-Stokes equations which we will derive in section 11 are the conservative forms of momentum for fluid mechanics. Using momentum instead of forces allows us to transfer many of the concepts we are familiar with when dealing with mass and heat to forces. This is why momentum is so useful, not only, but particularly in fluid mechanics.

9.5.1 Dynamic Viscosity

Returning to the example of the deck of cards we discussed earlier, we can observe that fluids show the same behavior. If the topmost fluid layer (which is in physical contact with the moved plane) is displaced, fluid layers underneath will be dragged along (see Fig. 9.4). The distance by which underlying layers will be dragged, as well as the speed observed, is dependent on properties inherent to a fluid. If the intermolecular forces are high within the fluid, which is the case, *e.g.*, in polar solvents (*e.g.*, water or saturated solutions or thick suspensions) the momentum is transferred effectively and the dragging effect observed is high. In fluids with low intermolecular forces (in gases) momentum transfer is lower. This ability for momentum transfer is referred to as the fluid’s *dynamic viscosity* or simply *viscosity*. The dynamic viscosity must be distinguished from the *kinematic viscosity*, which will be introduced later.

In a fluid with high viscosity, momentum is transferred effectively and underlying fluid layers will move at approximately the same velocity as the topmost layer. In a fluid with low viscosity, the momentum is transferred poorly, thus the velocity of underlying layers decreases quickly the further they are away from the topmost layer.

Fig. 9.4a shows the example of the fluid being sandwiched between two plates. The top plate is moved by an applied force F . This force is transferred onto the fluid *via* the area A of the top plate. The fluid therefore has an external shear stress τ applied to it. As discussed, the fluid will start moving along the direction of the acting force. Depending on the degree of internal friction, *i.e.*, the viscosity, the velocity profile that will be formed is different. If the viscosity is high the profile will be steep, *i.e.*, it will have a high gradient. If the viscosity is low, the profile will be shallow, *i.e.*, it will have a low gradient. Therefore, the viscosity is the proportionality term that links the applied shear stress τ with the gradient of the formed velocity profile $\frac{dv_x}{dz}$. It is therefore defined as

$$\eta = \frac{\tau}{\frac{dv_x}{dz}} \quad (\text{Eq. 9.2})$$

$$\tau = \eta \frac{dv_x}{dz} \quad (\text{Eq. 9.3})$$

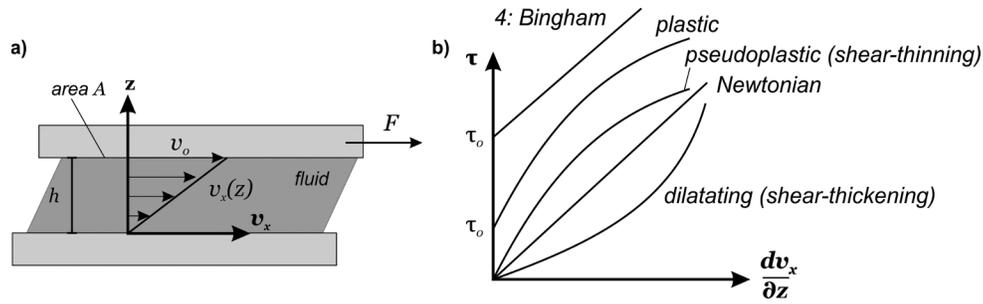


Fig. 9.4 Momentum transport in fluids. a) A fluid sandwiched between two plates is displaced once the top plate is moved. The fluid layer in direct contact with the top plate will move and, due to the momentum transport property the fluid, drag along underlying fluid layers. b) Depending on the fluid characteristic, analytical correlations can be found between the applied shear stress τ and the shear rate $\frac{dv_x}{dz}$.

Analytically, the viscosity η is a function of the shear force τ . It is measured by using a *viscosimeter*. There are various types of commercially available viscosimeters. Section 9.5.9 briefly describes the most important ones. Most of these systems allow plotting the shear force τ over the shear rate $\frac{dv_x}{dz}$ (see Eq. 9.3). After plotting this correlation, the viscosity can be seen graphically as the first derivative, *i.e.*, the slope of the plot (see Fig. 9.4b). Interestingly we can see that for different fluids, different functions are obtained. Usually they can be fitted with a polynomial according to

$$\tau = k \left(\frac{dv_x}{dz} \right)^n \quad (\text{Eq. 9.4})$$

Eq. 9.4 is referred to as the *Ostwald approximation*. We will discuss the five distinct behaviors of fluids displayed in Fig. 9.4b.

9.5.2 Newtonian Fluids

The first category is the *Newtonian fluid*. For Newtonian fluids, Eq. 9.4 simplifies to

$$\tau = k \frac{dv_x}{dz} = \eta \frac{dv_x}{dz} \quad (\text{Eq. 9.5})$$

So in Newtonian fluids, the viscosity is a constant and not dependent on the shear rate. If restricting a fluid mechanical problem to Newtonian fluids, the equations are significantly simplified. Typical Newtonian liquids are liquids in pure form and diluted solutions including water.

9.5.3 Bingham Fluids

*Bingham*¹ fluids are fluids that show Newtonian behavior above a certain minimum shear force τ_0

$$\tau = \eta \frac{dv_x}{dz} + \tau_0 \quad (\text{Eq. 9.6})$$

If shear forces below this threshold are applied, the fluid will behave like a solid. Above this threshold, the fluid will behave like a Newtonian fluid. Typical examples of Bingham fluids are gels and thick solutions, *e.g.*, ketchup, dough, tough paste, and blood.

9.5.4 Dilatant Fluids

Dilatant fluids are also referred to as *shear-thickening fluids*. The viscosity of these fluids will grow with increasing shear rate. For these fluids, the exponent Eq. 9.4 is $n > 1$. Typical examples of dilatant fluids are thick suspensions of particles in a liquid. If a shear rate is applied to these particles, they need to reorder in order to reduce the influence of the shear rate. By doing so, the overall shear force can be reduced. If the shear rate applied is small,

¹ Eugene Bingham was an American chemist who made important contributions to the study of momentum transport in fluids.

the particles have enough time to reorder. However, if a high shear rate is applied, the particles do not have the required time to reorganize and a significant shear force is built up.

A good example of a dilatant fluid is a suspension of corn starch in water. If such a suspension is compressed quickly by hand, the suspension will turn almost solid. If releasing the pressure, the suspension will flow freely again.

9.5.5 Pseudo-Plastic Fluids

Pseudo-plastic fluids are also referred to as *shear-thinning fluids*. The viscosity of these fluids will decrease with increasing shear rate. Therefore, for these fluids, the exponent Eq. 9.4 is $n < 1$. Typical examples for pseudo-plastic fluids are polymer solutions and similar solutions of high molecular weight substances. At low shear rates, these liquids will experience the formation of shear stress. The shear stress results in the reordering of the molecules in order to reduce the overall stress. This induction of a higher degree of order in the fluid reduces the shear stress and leads to the observed nonproportionality between the shear rate and the shear force.

9.5.6 Plastic Fluids

Plastic fluids behave similarly to pseudo-plastic fluids, but they have an initial threshold shear force τ_0 similar to Bingham fluids. Below this threshold value, they will behave like a solid. Above this threshold, they will behave like a pseudo-plastic fluid.

9.5.7 Time-Dependent Viscosity

In general, the viscosity is considered to be a time-invariant property. However, some fluids show increases (time-dependent shear-thickening) or decreases (time-dependent shear-thinning) in viscosity over time at constant shear rates (see Fig. 9.5). Fluids that show increases in viscosity over time are referred to as *thixotropic* fluids, with clay being a typical example. Fluids that show decreases in viscosity at constant shear rate over time are referred to as *rheopexic* fluids, with printer inks being a typical example.

9.5.8 Kinematic Viscosity

After introducing the dynamic viscosity, we will now turn to the *kinematic viscosity* ν which is used when normalizing the dynamic viscosity to the density of a fluid. It can be thought of as a “viscosity density”, *i.e.*, how much momentum a fluid can transfer per volume. It is defined as

$$\nu = \frac{\eta}{\rho} \quad (\text{Eq. 9.7})$$

The higher the kinematic viscosity, the more a fluid is able to transport momentum. Highly viscous fluids and liquids of low density are good candidates for excellent momentum transport properties.

Schmidt Number and Momentum Transport. The higher the kinematic viscosity of a fluid, the more efficient it can transport momentum. This property is often also referred to as the potential for *momentum diffusion* through the fluid. A fluid with fast momentum transport will account for more laminar flow conditions due to the fact that it balances the inertia forces of the in- and outflowing mass effectively. We have already

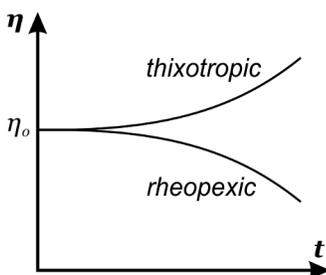


Fig. 9.5 Viscosity changes over time at constant shear rate $\frac{dv_x}{dz}$ in thixotropic (shear-thinning) and rheopexic (shear-thickening) fluids over time.

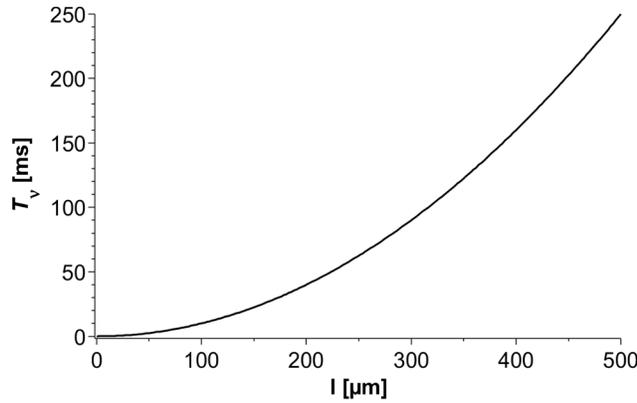


Fig. 9.6 Momentum transport in water as a function of the distance l .

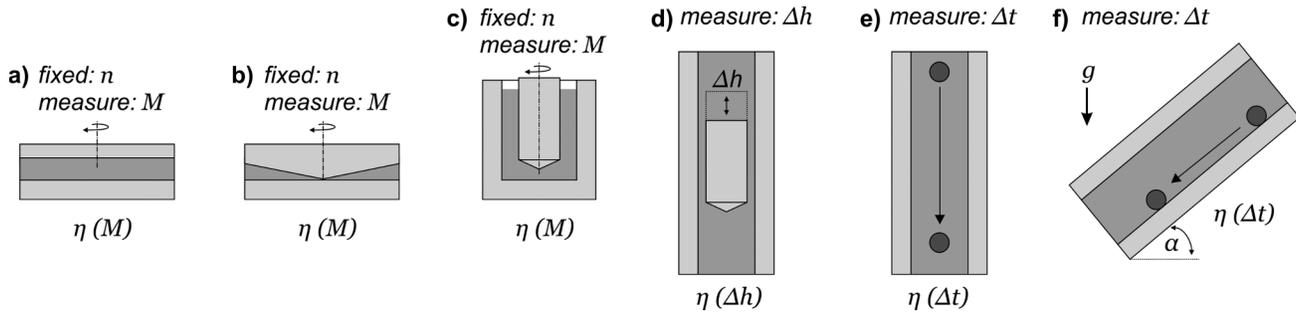


Fig. 9.7 Schematic setups of typical commercially available viscosimeter. a) Plate/plate, b) cone/plate, c) rotating piston, d) oscillating piston, e) falling ball, f) rolling ball viscosimeter.

derived the concepts of diffusion and found that the average diffusion length l is proportional to the square root of the product of time t and the diffusion coefficient D (see Eq. 6.89)

$$l^2 \propto tD \rightarrow t = \frac{l^2}{D}$$

Interpreting the kinematic viscosity ν as *momentum diffusion constant*, we find the time required for momentum to diffuse in a fluid as

$$t_\nu = \frac{l^2}{\nu}$$

In water it takes about 10 ms for momentum to diffuse over a length of $l = 100 \mu\text{m}$. The time increase with the square of the distance (see Fig. 9.6).

9.5.9 Viscosimeters

Viscosimeters (sometimes also referred to as *viscometers*) are instruments used to measure the viscosity of fluids. There are several types of commercially available viscosimeters (see Fig. 9.7). The most important ones will be explained in the following. All viscosimeters measure the shear force τ as a function of the shear rate $\frac{dv_x}{dz}$.

Plate/Plate Viscosimeter. The *plate/plate viscosimeter* or simply *plate viscosimeter* uses two plates one of which is rotating at a given rotation speed n while measuring the resulting torque M (see Fig. 9.7a). The forces rotation applies a known shear rate $\frac{dv_x}{dz}$ to the fluid. The torque measured is a function of the shear force τ .

Cone/Plate Viscosimeter. The *cone/plate viscosimeter* is similar to the plate/plate viscosimeter, but it uses a rotating cone instead of a flat plate (see Fig. 9.7b). The reason for using a cone instead of a plate is the following. As the rotation speed n is fixed, the shear rate will increase along the radius of the bottom plate as the angular rotation speed increases toward the outside. By increasing the height of the fluid above the plate, this shear rate increase can be compensated such that the shear rate does not change along the radius. This allows for significantly more precise measurements.

Again, the cone/plate viscosimeters apply a known shear rate $\frac{dv_x}{dz}$ to the fluid and measure the shear force τ by measurement of the torque.

Rotating Piston Viscosimeter. The *rotating piston viscosimeter* uses a piston inserted in a vessel of known size which is rotated at a constant speed n (see Fig. 9.7c). The rotation of the piston applies a known shear rate $\frac{dv_x}{dz}$ to the fluid, the torque measured is a direct measure for the shear force τ .

Oscillating Piston Viscosimeter. In the *oscillating piston viscosimeter*, a piston is immersed in a liquid and moved in a magnetic field (see Fig. 9.7d). The amplitude of the oscillation is a measure for the induced shear force τ that slows the motion of the piston. By measuring the amplitude, the viscosity can be derived.

Falling Ball Viscosimeter. The *falling ball viscosimeter* measures the viscosity as a function of the travel time a falling ball requires to travel a given distance in the fluid while in free fall (see Fig. 9.7e). This concept balances the gravimetric force of the sphere against the *Stokes drag*, *i.e.*, the resistance a spherical object will experience while moving in a viscous ambience. Falling ball viscosimeters do not allow the direct derivation of the shear rate $\frac{dv_x}{dz}$ over the shear force τ . Rather they allow measuring the viscosity directly.

Rolling Ball Viscosimeter. The *rolling ball viscosimeter* is a light modification of the falling ball viscosimeter (see Fig. 9.7f). Instead of being in free fall, the ball is rolling on an inclined plane being slowed by the fluid surrounding it. Compared to the falling ball viscosimeter, the rolling ball viscosimeter allows modulation of the gravimetric force and thus, the acceleration of the ball in order to allow measurement of liquids with very small viscosity values. In such liquids, falling ball viscosimeters can usually not be applied due to the very short measurement interval.

Ostwald Viscosimeter. The *Ostwald¹ viscosimeter* allows measuring the kinematic viscosity ν using the hydraulic resistance of a circular tube. From the pressure drop, the viscosity can be calculated according to the law of Hagen-Poiseuille. For this we need Eq. 16.22 which gives the flow rate for liquid flow through a circular tube as

$$Q = \frac{\Delta V}{\Delta t} = \frac{\pi R^4}{8\eta} \frac{dp}{dx}$$

which we can rewrite to

$$\eta = \frac{\pi R^4 \Delta t}{8\Delta V} \frac{dp}{dx} \quad (\text{Eq. 9.8})$$

The Ostwald viscosimeter is depicted in Fig. 9.8. It essentially is a U-shaped tube with two larger reservoirs termed *A* (the larger one) and *B* (the smaller one). A liquid is filled into the tube such that reservoir *A* is completely filled. Suction is applied on the right-hand tube of the viscosimeter and the liquid is pulled up beyond reservoir *B* until it reaches the upper-level mark. A stopwatch is started and the suction released. The liquid column will reequilibrate. The time Δt it takes the liquid to pass below the lower-level mark is measured. Please note that the tube has its smallest diameter in the section right underneath reservoir *B*. This section will determine the flow rate because it is the section with the smallest diameter and thus, the highest hydraulic resistance.

If you study Eq. 9.8, you will see that we now have the value Δt . It is difficult to measure some of the other values which is why the experiment is usually run with a second liquid of known viscosity value η_0 for which the value Δt_0 is obtained. We are then able to find the relative value of the unknown viscosity η that cancels out most of the constants of the equation. We only need to account for the differences in pressure drops $\frac{dp}{dx}$. The pressure drop is due to the gravimetric height difference which is only given by the two-level marks and thus equal for both experiments. The pressure is calculated as the weight per area as

$$p = \frac{mgh}{\pi r^2}$$

¹ Wilhelm Ostwald was a German-Russian chemist who invented, among others, the *Ostwald viscosimeter*. Ostwald is credited as one of the first physical chemist. He received the *Noble Prize* in chemistry in 1909 for his work on catalysis, chemical kinetics, and the study of the equilibrium of chemical reactions.

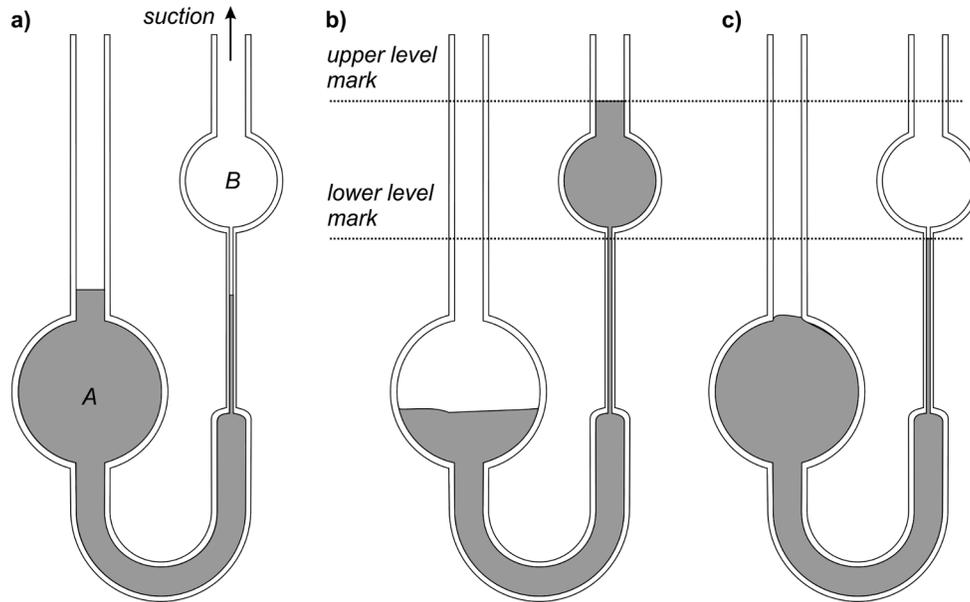


Fig. 9.8 Setup of the Ostwald viscosimeter. a) The U-shaped tube is filled with the liquid for which the viscosity is to be measured. The liquid level is chosen such that reservoir A is completely filled. b) The liquid column is pulled up to the level of the upper level mark by suction on the right-hand tube. c) Releasing the suction will make the liquid column reequilibrate. The time it takes the liquid to flow back to its original level is dependent on the hydraulic resistance of the tube section with the smallest diameter. The time is also dependent on the viscosity of the liquid.

from which we calculate the pressure difference as

$$\begin{aligned} \frac{dp}{dx} &= \frac{m g h}{\pi r^2} \\ &= \frac{h}{m g} \\ &= \frac{V \rho g}{\pi r^2} \end{aligned} \tag{Eq. 9.9}$$

Using Eq. 9.8 and Eq. 9.9 we can then find the relative viscosity to be

$$\begin{aligned} \frac{\eta}{\eta_0} &= \frac{\Delta t}{\Delta t_0} \frac{\frac{dp}{dx}}{\left(\frac{dp}{dx}\right)_0} \\ &= \frac{\Delta t \rho}{\Delta t_0 \rho_0} \end{aligned}$$

which we can solve for the unknown viscosity value finding

$$\eta = \eta_0 \frac{\Delta t \rho}{\Delta t_0 \rho_0} \tag{Eq. 9.10}$$

The Ostwald viscosimeter yields very exact results. However, care has to be taken to note the temperature during the experiment, as the viscosity is a function of the temperature.

9.6 HEAT TRANSPORT

Fluids are able to transport heat. The potential to transport heat is an intrinsic property of the fluid and is not dependent on the flow condition. Heat transport is analytically described by Fourier's law of heat conduction (see section 6.9.2).

So as an example, Fig. 9.9 shows the pressure-driven flow profile in a thin infinitesimally extended slit (see section 16.2). If the top plate is heated, there will be heat conduction in addition to momentum transfer in the

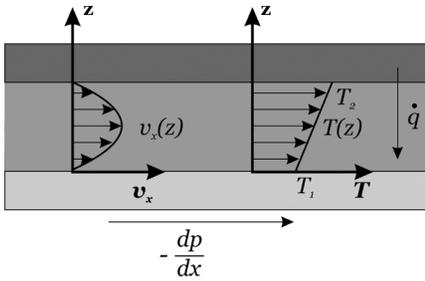


Fig. 9.9 A Hagen-Poiseuille flow in a slit channel with a heated top plate. Besides momentum transport (which is the reason for the formation of the flow profile) there is also heat flow in the fluid. A temperature profile will develop which is independent of the flow profile.

fluid. The temperature profile develops over time, but its development is pretty much independent of the flow profile. This is due to the fact that the fluid is treated as a continuum and therefore is an analogy to the thin plate model used to derive Fourier's law of heat conduction (see section 6.9.2).

Please note that in the most general form, Fourier's law of heat conduction (see Eq. 6.71) does not take into account warming of the fluid (which of course happens). These equations are only valid for stationary heat conduction. Of course, warming of the fluid needs to be taken into account which expands the Fourier law to include changes of enthalpy of the fluid. In this case, Eq. 6.82 must be applied (see section 6.9.2.2). This equation is contained in the conservation of energy equation that we will derive in section 12. We will mainly apply the stationary version of Fourier's law of heat conduction in the derivation of the fundamental equations of fluid mechanics because we mainly consider the heat conduction while taking account of the change of enthalpy in the energy conservation equation.

9.7 MASS TRANSPORT

The last transport property we need to consider is diffusion. Diffusion is the nonconvective mass transport in a fluid. Being nonconvective, diffusion will also occur when the fluid is at rest. Diffusion is driven by concentration gradients and is described analytically by Fick's laws of diffusion (see Eq. 6.90). The fundamental laws of diffusion are derived in section 6.9.3.

9.8 BOUNDARY CONDITIONS

9.8.1 Introduction

As we will see, we will frequently require boundary conditions that define the properties of our fluid continuum at the boundary of the domain. There are two approaches to understanding why boundary conditions are required.

The first is the logical one. If we want to describe how a fluid behaves inside, *e.g.*, a microfluidic channel, we obviously need to know how the boundary of this microfluidic channel behaves. If this boundary is moving, *e.g.*, because the channel has a movable cover, the fluid mechanics inside of the system will be different. A moving cover alone is able to move fluids due to their capability for momentum transport (which is the nature of viscosity). As an example, see the Couette flow, discussed in section 15.2.

The second approach to understanding why we need boundary conditions is purely mathematical. If the underlying fluid mechanics is described in the form of, *e.g.*, a second-order ODE, then obviously we require integration constants. These are provided by the boundary conditions.

Boundary conditions apply to several field variables, the most important ones are the *velocity boundary conditions*, the *shear stress boundary conditions*, the *temperature boundary conditions*, and the *pressure boundary conditions*.

9.8.2 Velocity

For the flow velocity, there are two main types of boundary conditions: no-slip and slip boundary conditions (see Fig. 9.10).

No-Slip Boundary Condition. The *no-slip boundary condition* or *no-velocity-offset boundary condition* assumes that the speed of the fluid layer in direct contact with the boundary is identical to the velocity of this boundary. There is no relative movement between the boundary and this fluid layer, therefore there is no slip. The

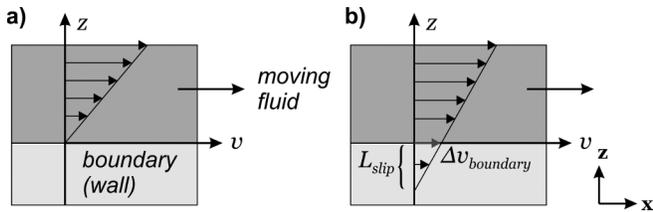


Fig. 9.10 No-slip (a) and slip (b) boundary conditions. In no-slip boundary conditions, the speed of the fluid at the wall is supposed to be zero, whereas in slip boundary conditions there is relative movement between the wall and the fluid. Most fluid mechanical problems on liquids can be considered using no-slip boundary conditions if the characteristic length scale L_{char} is bigger than 300 nm.

slip boundary condition or *velocity-offset boundary condition* assumes a discontinuity in the velocity function, *i.e.*, a relative movement between the fluid and the boundary, therefore there is slip. The hypothetical distance inside of the boundary at which the velocity function would effectively reach the velocity of the boundary is referred to as *slip length* L_s . The discrepancy between the velocity of the last layer of the fluid field and velocity at the boundary can be estimated to be

$$\Delta v_{\text{boundary}} = \left. \frac{\partial v_x}{\partial z} \right|_{\text{boundary}} L_s$$

The slip length is only dependent on the pairing of the fluid and the solid and can thus, theoretically, be determined experimentally. In section 9.9.2 we introduce the Knudsen number which allows a rough numerical approximation of when to use which of these two conditions. In general, the no-slip boundary condition can be applied in most applications in microfluidics when dealing with solid boundaries and incompressible fluids at characteristic length scales above 300 nm.

Slip Boundary Condition. Whenever the no-slip boundary condition cannot be applied, slip has to be considered. In these scenarios, the velocity at the boundary is nonzero. As stated, this case is rare in microfluidics.

9.8.3 Shear Stress

Obviously, if the shear stress at a specific boundary of the system is known, it can also be used as a boundary condition.

Free Surface Condition. However, the most important boundary condition for the shear stress is the *free surface boundary condition*. As stated, a fluid is able to transport momentum, which gives rise to its viscosity. This momentum transport results in the formation of shear stress in the fluid. The individual molecule layers inside of the fluid transport the momentum from one layer to the next. However, the last layer of molecules of the fluid's surface, the *free surface* cannot transport the momentum onwards, as the boundary is discontinuous. Assuming the next layer would be a gas layer (*e.g.*, air), the momentum cannot be transported onwards efficiently. This is why, at this last molecule layer, the shear stress τ must be 0. This is referred to as the *free surface condition*.

9.8.4 Temperature

The temperature profile is the second important profile to consider besides the velocity. Again, there are two potential scenarios. In case of the *no-temperature-offset boundary condition*, the temperature function is assumed to be continuous, and the temperature of the fluid layer directly above the boundary is assumed to be identical to the temperature of the boundary. In the case of the *temperature-offset boundary condition*, the temperature profile is noncontinuous and there is an offset in temperature of the last layer of the flow field compared to the boundary.

In general, given the high surface-to-volume ratios, most microfluidic systems using incompressible fluids can be treated using no-temperature-offset boundary conditions. If applying the no-slip boundary condition, it is usually also possible to apply the no-temperature-offset boundary conditions. Thus a more precise account of whether or not it is possible to apply these boundary conditions can be derived from the Knudsen number (see section 9.9.2).

9.8.5 Pressure

The last important field variable to consider is the pressure. Again, there are two approaches to how to treat the boundary condition. The *no-pressure-offset boundary condition* assumes the pressure in the boundary to

be identical to the normal component of the stress tensor in the fluid layer directly above the boundary. The *pressure-offset boundary condition* assumes an offset in the pressure function between the last fluid layer of the flow field and the boundary. In a microfluidic system, the no-pressure-offset boundary condition can usually be applied. We will use this assumption when discussing the case of pressure being applied to a frictionless flow field in section 11.6.1.

9.9 DIMENSIONLESS NUMBERS

As we will see, numerous problems in fluid mechanics can be made significantly easier if certain effects in a flow field can be neglected. As an example, in a fast-flowing fluid field, the effect of convection is usually dominating compared to the effect of diffusion. In these cases, the diffusion terms can be neglected, which renders the overall fluid mechanical problem significantly simpler to handle.

In order to judge whether or not certain effects can, in fact, be neglected, a selection of *dimensionless numbers* are used. The property *dimensionless* refers to the fact that they can be applied to a wide range of fluid mechanical problems that may differ significantly in their scale. The numbers are chosen such that they exploit similar effects in fluid mechanics. This means that even if two systems do not have the same length scales, the fluid mechanics in the system are the same because certain properties of the system are *fluid mechanically similar*. As an example, the Prandtl number Pr (see section 9.9.3) relates the capability for momentum transport of a liquid to its capability for heat transport. As can be seen from Eq. 9.13, this number is an intrinsic property of the fluid and therefore independent of the fluid mechanical problem. Therefore two systems, even though they feature different length scales, will behave identically when considering effects originating from the Prandtl number.

However, in many applications there will be at least one length scale involved. A common example is the Reynolds number Re which relates inertia to the viscous forces in a fluid mechanical system. It allows judging whether or not a flow is laminar. Laminar flows are stabilized by the viscous forces of the flow field that do not allow the flow to accelerate as far as to become instable. Therefore, in laminar flows, the viscous terms dominate over the inertial terms. As can be imagined, this is not only due to intrinsic properties of the fluid in the flow, but also dependent on the velocity of the flow and the influence of the ambience which may stabilize and order the flow. In a microfluidic channel, the channel walls will have a significant stabilizing effect upon the flow field as they force the flow to obtain a certain degree of order. If the channel becomes wider, this influence decreases and the flow in the center of the channel may become instable eventually. Here, the fluid mechanical system has also an influential effect and thus must be considered as well. In these cases, a *characteristic length scale* L_{char} is used which expresses this influence. In the case of the Reynolds number, the smallest channel dimension, *i.e.*, either the channel height or the channel width is usually chosen as characteristic length scale, as this dimension will exert the highest stabilizing influence.

In the following, the most important dimensionless numbers will be introduced with a short explanation of their usage and interpretation. A summary can be found in Tab. 9.2.

9.9.1 Mach Number

The first dimensionless number we will discuss is the *Mach number* which is defined as

$$M = \frac{v}{c_s} = \frac{\text{velocity of flow}}{\text{speed of sound}} \quad (\text{Eq. 9.11})$$

The Mach puts in relation the current flow speed v and the speed of sound c_s . If a gas flow is stationary and isothermal and not flowing at velocities above 30% of the speed of sound, the gas can be considered to be incompressible, whereas above this threshold, it is considered to be compressible. Considering a gas as being incompressible makes the underlying fluid mechanics significantly simpler. Checking the Mach number prior to performing calculations on the fluid mechanics of a gas flow can therefore save a lot of computational and analytical effort.

9.9.2 Knudsen Number

The *Knudsen number* is an important dimensionless quantity which allows characterizing the boundary conditions of a fluid flow. It is defined as

$$Kn = \frac{\lambda}{L_{char}} = \frac{\text{mean free path length}}{\text{char. length}} \quad (\text{Eq. 9.12})$$

Tab. 9.2 *The most important dimensionless numbers used in fluid mechanics*

Number	Equation	Comment
Bond ^a	$Bo = \frac{\rho * g * L_{char}^2}{\gamma} = Eo = \frac{\text{buoyancy forces}}{\text{surface tension}}$	relates buoyancy (gravitational) forces to surface tension
Capillary	$Ca = \frac{\eta * v}{\gamma} = \frac{\nu * \rho * v}{\gamma} = \frac{\text{viscous forces}}{\text{surface tension}}$	relates viscous forces to surface tension
Eckert ^b	$Ec = \frac{v^2}{c_p * T_{char}} = \frac{\eta}{c_p * \lambda} = \frac{\text{kinetic energy}}{\text{enthalpy}}$	relates kinetic energy to enthalpy
Eötvös ^c	$Eo = \frac{\rho * g * L_{char}^2}{\gamma} = Bo = \frac{\text{buoyancy forces}}{\text{surface tension}}$	relates buoyancy (gravitational) forces to surface tension
Froude ^d	$Fr = \frac{v^2}{g L_{char}} = \frac{\text{inertia forces}}{\text{gravity forces}}$	relates inertia forces to gravitational forces
Knudsen ^e	$Kn = \frac{\lambda}{L_{char}} = \frac{\text{mean free path length}}{\text{char. length}}$	measure for the validity of the continuum hypothesis; for $Kn \approx 1$ the length scale of the problem is near the mean free path of the molecules and the continuum hypothesis is not valid
Lewis ^f	$Le = \frac{D}{\alpha} = \frac{\text{mass transport}}{\text{heat transport}}$	relates convective transport to diffusive mass transport
Mach ^g	$M = \frac{v}{c_s} = \frac{\text{velocity of flow}}{\text{speed of sound}}$	measure for the compressibility of a gas; for $M < 0.3$ a gas is considered incompressible; for $M > 0.3$ the gas is considered compressible
Marangoni ^h	$Ma = \frac{d\gamma}{dT} \frac{L_{char} * \Delta T}{\eta * \alpha} = \frac{\text{Marangoni forces}}{\text{viscous forces}}$	relates forces due to thermal (temperature) gradients (Marangoni forces) to viscous forces
Ohnesorge ⁱ	$Oh = \frac{\text{viscous forces}}{\text{inertia forces} \cdot \text{surface tension}}$	relates viscous forces to the product of inertia and surface tension forces
Péclet ^j	$Pe = \frac{v L_{char}}{D} = \frac{\text{convection transport}}{\text{diffusion transport}}$	relates convective to diffusive mass transport
	$Pe = \frac{v L_{char}}{\lambda} = \frac{\text{convection transport}}{\text{heat transport}}$	relates convective to heat transport
Prandtl ^k	$Pr = \frac{\nu}{\alpha} = \frac{\eta}{\rho * \alpha} = \frac{\eta * c_p}{\lambda} = \frac{\text{momentum transport}}{\text{heat transport}}$	relates momentum transport to heat transport
Reynolds	$Re = \frac{\rho * v * L_{char}}{\eta} = \frac{v * L_{char}}{\nu} = \frac{\text{inertia forces}}{\text{viscous forces}}$	measure for the turbulence of a flow; for $Re \ll 1$ the viscous effects dominate (strictly laminar flow); for $Re \gg 1$ the inertia effects dominate (turbulent flow)
Schmidt ^l	$Sc = \frac{\nu}{D} = \frac{\eta}{\rho * D} = \frac{\text{momentum transport}}{\text{mass transport}}$	relates momentum transport to mass transport
Weber ^m	$We = \frac{\rho * v^2 * L_{char}}{\gamma} = \frac{\text{inertia forces}}{\text{surface tension}}$	relates inertia forces to surface tension

^a named after English physicist Wilfrid Bond
^b named after Hungarian engineer Ernst Eckert
^c named after Hungarian physicist Loránd Eötvös
^d named after English engineer William Froude
^e named after Danish physicist Martin Knudsen
^f named after American engineer Warren Lewis
^g named after Austrian physicist Ernst Mach
^h named after Italian physicist Carlo Marangoni
ⁱ named after German physicist Wolfgang von Ohnesorge
^j named after French physicist Eugène Péclet
^k named after German physicist Ludwig Prandtl
^l named after German engineer Ernst Schmidt
^m named after German engineer Moritz Weber

It puts the mean free path λ (see Eq. 6.11) and a characteristic length scale L_{char} of the fluid mechanical system at hand in perspective. It gives a numerical account of whether or not the continuum hypothesis (see section 9.3.2) can be applied. Remember that the mean free path is the length that a molecule can travel before encountering a collision event with a second molecule (see section 6.4.2). If the length scale of the fluidic system is in the same range as the mean free path, *i.e.*, $\text{Kn} = 1$, the fluid cannot be treated as a continuum.

The Knudsen number is of particular interest when assessing the boundary of fluid flows. Usually, the flow at the boundary of a flow field, *i.e.*, the channel walls are fixed in space and the liquid directly in contact is considered not to move. This is referred to as the *no-slip boundary condition*, *i.e.*, there is no relative movement (slip) between the wall and the fluid layer in direct contact with the wall (see Fig. 9.10a). This is the case if the characteristic length of the fluidic system is significantly bigger than the mean free path. The threshold value assumed is usually $\text{Kn} < 0.001$. If the characteristic dimension of the fluid system is decreased to a value near the mean free path, there will be relative movement between the wall and the fluid layer directly in contact (see Fig. 9.10b). The characteristic range for the Knudsen number is $0.001 < \text{Kn} < 0.1$. In this case, the *slip boundary condition* is assumed. For Knudsen numbers above 0.1, there is no continuum and most of the gas flow must be characterized using statistical methods.

One thing to keep in mind is that for gases, the compressibility must be considered. If the pressure is increased, the average mean free path is decreased (see section 9.3.4 and section 9.3.5) and thus, the Knudsen number reduces. As liquids are incompressible, the mean free path can be considered a constant. A good approach for water is to set $\lambda \approx 0.31 \text{ nm}$ (see section 9.3.3). Therefore, in microfluidic channels with characteristic length scales of more than 300 nm, we can safely assume no-slip boundary conditions.

9.9.3 Prandtl Number

The *Prandtl number* is a dimensionless quantity that puts the viscosity of a fluid in correlation with the thermal conductivity. It therefore assesses the relation between momentum transport and thermal transport capacity of a fluid. It is defined as

$$\text{Pr} = \frac{\nu}{\alpha} = \frac{\eta}{\rho \alpha} = \frac{\eta c_p}{\lambda} = \frac{\text{momentum transport}}{\text{heat transport}} \quad (\text{Eq. 9.13})$$

where we have used the *thermal diffusivity* α which is defined as

$$\alpha = \frac{\lambda}{\rho c_p} \quad (\text{Eq. 9.14})$$

The Prandtl number is an example of a dimensionless number that is an intrinsic property of a fluid. Fluids with small Prandtl numbers are free-flowing liquids with high thermal conductivity and are therefore a good choice for heat conducting liquids. Tab. 9.3 lists Prandtl values for selected fluids. As can be seen, liquid metals are very good heat transfer liquids. Interestingly, air is a decent heat transfer liquid as well, whereas typical organic solvents are not. With increasing viscosity, the momentum transport dominates over the heat transport, which makes these liquids a bad choice for heat conduction.

Usually, the Prandtl number is assumed to be around 0.7 for gases and around 6.9 for water.

9.9.4 Eckert Number

The *Eckert number* relates the kinetic energy to the enthalpy of a fluid. It is given by

$$\text{Ec} = \frac{v^2}{c_p T_{\text{char}}} = \frac{\eta}{c_p \lambda} = \frac{\text{kinetic energy}}{\text{enthalpy}} \quad (\text{Eq. 9.15})$$

The Eckert number is used to characterize the influence of self-heating of a fluid as a consequence of dissipation effects. At high flow velocities, the temperature profile in a fluidic system is not just dominated by the temperature gradients that are present in the system, but also by effects of dissipation due to internal friction of the fluid. This will result in self-heating and thus in a change of the temperature profile. The Eckert number allows judging if the effects of self-heating due to dissipation can be neglected ($\text{Ec} \ll 1$) or not.

9.9.5 Schmidt Number

The *Schmidt number* is a dimensionless quantity that puts the viscosity of a fluid in correlation with its diffusion coefficient. Similar to the Prandtl number, which correlates momentum transport and thermal transport properties of a fluid, the Schmidt number puts momentum transport and mass transport in correlation. It is defined as

Tab. 9.3 Prandtl values for some selected fluids. Data calculated from [4]

Fluid	Temp. °C	Ther. conduct. λ W m ⁻¹ K ⁻¹	Viscosity η mPa s	Heat capacity c_p J kg ⁻¹ K ⁻¹	Prandtl number
sodium	100	60	0.542	1.225	0.01
mercury	25	8.250	1.526	0.140	0.03
air	30	0.026	0.019	1.010	0.72
carbon disulfide	25	0.149	0.352	1	2.36
chloromethane	25	0.117	0.537	0.960	4.41
methanol	25	0.200	0.544	2.510	6.83
water	25	0.607	1.002	4.182	6.90
toluene	25	0.131	0.560	1.700	7.26
ethanol	25	0.169	1.074	2.840	18.05
argon	30	0.018	22.900	0.018	22.77
krypton	30	0.010	25.600	0.250	673.68
xenon	30	0.006	23.200	0.160	674.91
glycerol	25	0.292	934	2.380	7 612.74

Tab. 9.4 A selection of Schmidt and Lewis values of commonly used fluids in water. Calculated from [4]

Fluid	Molecular diffusion coefficient D 10 ⁻⁵ cm ² s ⁻¹	Thermal diffusivity α 10 ⁻³ m ² s ⁻¹	Schmidt number	Lewis number
methanol	1.28	0.100	537.02	127.13
toluene	0.85	0.089	759.98	95.55
ethanol	1.24	0.075	1 097.34	164.47
glycerol	1.06	0.097	698 590.40	108.97
water		0.145	759.98	95.55

$$\text{Sc} = \frac{\nu}{D} = \frac{\eta}{\rho D} = \frac{\text{momentum transport}}{\text{mass transport}} \quad (\text{Eq. 9.16})$$

The Schmidt number is a material property and not dependent on the flow. A selection of Schmidt values for some commonly encountered fluids is given in Tab. 9.4.

9.9.6 Lewis Number

The *Lewis number* puts in correlation the mass diffusion and the thermal conductivity of a fluid. Similar to the Prandtl number, which correlates momentum transport and thermal transport properties of a fluid, and similar to the Schmidt number, which correlates momentum transport and mass transport of a fluid, the Lewis number correlates mass transport to thermal transport properties of the fluid. It is defined as

$$\text{Le} = \frac{D}{\alpha} = \frac{\text{mass transport}}{\text{heat transport}} \quad (\text{Eq. 9.17})$$

Similar to the Prandtl and the Schmidt numbers, the Lewis number is a material constant. A selection of Lewis values for some commonly encountered fluids is given in Tab. 9.4.

9.9.7 Péclet Number

The *Péclet number* puts convective and diffusive transport phenomena in correlation. It is related to both the Prandtl number (see section 9.9.3), which correlates momentum transport and thermal transport, as well as the Reynolds (see section 9.9.8) number, which correlates inertia and viscous forces. The Péclet number is defined as

$$\text{Pe} = \frac{vL_{\text{char}}}{D} = \frac{\text{convection transport}}{\text{diffusion transport}} \quad (\text{Eq. 9.18})$$

There is a second commonly used definition of the Péclet number that relates convective transport to heat transport. It is defined as

$$\text{Pe} = \frac{vL_{\text{char}}}{\lambda} = \frac{\text{convection transport}}{\text{heat transport}} \quad (\text{Eq. 9.19})$$

Similar to the Reynolds number, the Péclet number is not a material constant, as it depends both on the velocity of the flow field and a characteristic length L_{char} of the system. Section 11.8.5 discusses the application of the Péclet number in dimensional analysis for fluid mechanical problems involving convection and diffusion.

9.9.8 Reynolds Number

The *Reynolds number* is one of the most important dimensionless quantities in microfluidics. It correlates the inertia forces to the viscous forces. The Reynolds number was first described by Reynolds in 1883 [5], although others have used the quantity before, *e.g.*, Stokes [6]. It is defined as

$$\text{Re} = \frac{\rho v L_{\text{char}}}{\eta} = \frac{v L_{\text{char}}}{\nu} = \frac{\text{inertia forces}}{\text{viscous forces}} = \frac{\text{Pe}}{\text{Sc}} \quad (\text{Eq. 9.20})$$

The Reynolds number is important for describing the transport properties of a fluid or a particle moving in a fluid. As an example, for very small organism, *e.g.*, bacteria, the Reynolds number is very small, typically in the range of 1×10^{-6} . Given the small dimensions, these objects do not have a significant inertia and are thus mainly driven by the viscous forces of the fluid. For such objects, a fluid would feel significantly more rigid, *i.e.*, it would be difficult for a bacteria to force a path through a moving fluid not following the streamlines. As the objects grow larger, their inertia starts to dominate over the viscous forces. For most fish, the Reynolds number is in the range of 1×10^5 , for a human it is in the range of 1×10^6 . At higher Reynolds numbers, an object is able to force its way through a flow field even across the streamlines. A good example is a large vessel or ship (with Reynolds numbers in the range of 1×10^9) compared to a folded origami or paper boat: the large vessel can force its way through the current and the waves, whereas the light paper boat would not be able to do so. Rather it has to stay with the streamline and will be dragged along.

As the Reynolds number is so important for microfluidics, we will detail its meaning and application when discussing the concept of dimensional analysis in section 11.8.3.

9.9.9 Eötvös or Bond Number

The *Eötvös number* and the *Bond number* are the same dimensionless quantity. They relate the buoyancy forces of a fluid in a system to the fluid's surface tension. They are defined as

$$\text{Eo} = \frac{\rho g L_{\text{char}}^2}{\gamma} = \text{Bo} = \frac{\text{buoyancy forces}}{\text{surface tension}} \quad (\text{Eq. 9.21})$$

The Eötvös, *i.e.*, the Bond number allow discriminating if in a given system, the buoyancy forces, *i.e.*, the forces due to the gravity can be neglected with respect to the forces originating from surface tension. Typical examples where this is the case are fluidic systems that handle small droplets suspended in an inert fluid, *e.g.*, in droplet microfluidics or if handling droplets on open surfaces, *e.g.*, in digital microfluidics.

Very often, a difference in density $\Delta\rho$ may be used instead of only a density. In these applications, the fluid's shape while embedded in a second fluid is generally discussed. Both numbers give a measure of how strongly a fluid is deformed by the pressure or density gradients and how well this fluid can sustain these pressure difference by means of its surface tension, *i.e.*, the strength of the cohesive surface. In such applications, the characteristic length L_{char} may be the diameter of a droplet. Similar balances of forces are found, *e.g.*, in the Young-Laplace equation (see section 20.1.9).

9.9.10 Capillary Number

The *capillary number* is a dimensionless quantity that relates the viscous forces in a system to the surface tension forces. It is defined as

$$\text{Ca} = \frac{\eta v}{\gamma} = \frac{\nu \rho v}{\gamma} = \frac{\text{viscous forces}}{\text{surface tension}} \quad (\text{Eq. 9.22})$$

The capillary number is used whenever the forces resulting from fluid motion are to be compared to the forces resulting from surface tension. This is the case if a liquid is moved across a second fluid layer, *e.g.*, a gas or an immiscible second liquid. A good visual example for these effects are droplets suspended in an inert liquid in droplet microfluidics. The viscous forces of the surrounding inert liquid may deform the droplets due to, *e.g.*, local increases in pressure due to variations in the flow conditions. However, the interface tension forces between the two liquids are usually significantly higher, in which case the droplet may be locally deformed, but not destroyed. In this case $Ca \ll 1$.

9.9.11 Froude Number

The *Froude number* relates the inertia forces in a system to the effects due to gravity. It is defined as

$$Fr = \frac{v^2}{gL_{\text{char}}} = \frac{\text{inertia forces}}{\text{gravity forces}} \quad (\text{Eq. 9.23})$$

The Froude number allows assessing whether or not in a fluid mechanical problem, the inertia terms dominate over effects of gravity. If this is the case, gravity can be neglected. A very prominent example of the importance of the Froude number can be seen in the example of the falling jet stream, *i.e.*, a fluid cylinder which is accelerated under the effect of gravity. As shown in section 23.3, gravitational effects can be neglected if the liquid cylinder has a high initial velocity. In these cases, the Froude number is very high.

9.9.12 Weber Number

Similar to the Froude number Fr , which relates the inertia forces to the gravitational forces, the *Weber number* relates the inertia forces to the forces resulting from surface tension. It is defined as

$$We = \frac{\rho v^2 L_{\text{char}}}{\gamma} = \frac{\text{inertia forces}}{\text{surface tension}} \quad (\text{Eq. 9.24})$$

Similar to the Froude number, the Weber number is used, *e.g.*, during the derivation of the fluid mechanics of the fluid jet. Depending on the initial velocity of the fluid in this problem, the effects of surface tension may be entirely neglected (see section 23.3). In these cases, the Weber number is very high.

9.9.13 Ohnesorge Number

The *Ohnesorge number* relates viscous forces to the product of the inertia and the surface tension forces. The Ohnesorge number can be expressed by the Weber number (see section 9.9.12) and the Reynolds number (see section 9.9.8). It is defined as

$$Oh = \frac{\text{viscous forces}}{\text{inertia forces} \cdot \text{surface tension}} = \frac{\sqrt{We}}{Re} \quad (\text{Eq. 9.25})$$

The Ohnesorge number allows assessing if in a given fluid mechanical problem, the inertia terms can be neglected compared to the inertia and surface tension forces. This is relevant, *e.g.*, for falling fluid jets in which the viscous forces can often be neglected, compared to surface tension and inertia (see section 23).

9.9.14 Marangoni Number

The *Marangoni number* relates the forces due to temperature gradients (see section 20.4) to the viscous forces. It is defined as

$$Ma = \frac{d\gamma}{dT} \frac{L_{\text{char}} \Delta T}{\eta \alpha} = \frac{\text{Marangoni forces}}{\text{viscous forces}} \quad (\text{Eq. 9.26})$$

The Marangoni number is important when considering boundary conditions of fluidic system that may (undesirably) induce fluid flow due to inhomogeneous heating. This effect is often referred to as *thermal creep* and may, especially for gas flows or fluids with low viscosity, contribute significantly to the overall fluid movement in the technical system.

9.9.15 Aspect Ratio

The *aspect ratio* r is a term commonly used in MEMS and usually refers to the ratio of a critical lateral dimension to the height of a microstructure. In most applications the aspect ratio is defined as the height h of a structure over the width w as

$$r = \frac{h}{w} \quad (\text{Eq. 9.27})$$

Usually, the higher the aspect ratio, the more challenging it is to manufacture the structure. As an example, a line structure of width $100\ \mu\text{m}$ with an aspect ratio of 1 would be $100\ \mu\text{m}$ width in height. Such dimensions can be achieved by suitable resists and careful manufacturing. If the aspect ratio is increased to 10, the structure would have an overall height of 1 mm. This would already be very challenging in terms of mechanical stability during manufacturing.

In microfluidics we usually describe a channel by means of its aspect ratio. Typically, aspect ratios are small in microfluidic with a typical microfluidic channel being $100\ \mu\text{m}$ wide and $50\ \mu\text{m}$ deep, we end up with an aspect ratio of 0.5. Higher aspect ratio channels are difficult to manufacture, as the number of methods for creating very thin, but very deep channels is quite limited.

9.10 SUMMARY

In this section we have introduced fluids and their most important properties. These concepts are important to understand the conservative laws of fluid mechanics, which we will discuss in the following sections. The transport property of fluids, *i.e.*, their ability to transport, *e.g.*, momentum and heat give rise to the fundamental equations required to understand and describe flow fields. In this section, we have also introduced the concept of dimensionless numbers that we will use frequently in order to simplify the equations we have to solve.

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